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ON THE BERRY-ESSEEN THEOREM FOR SIMPLE LINEAR RANK STATISTICS.(U)
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ON THE BERRY-ESSEEN THEOREM FOR
SIMPLE LINEAR RANK STATISTICS

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ABSTRACT

ON THE BERRY-ESSEEN THEOREM FOR
SIMPLE LINEAR RANK STATISTICS

The rate of convergence $O(N^{-\frac{1}{2}+\delta})$ for any $\delta > 0$ is established for two theorems of Hájek (1968) on asymptotic normality of simple linear rank statistics. These pertain to smooth and bounded scores, arbitrary regression constants, and broad conditions on the distributions of individual observations. The results parallel those of Bergström and Puri (1977), which appeared in print just as this paper was completed. Whereas Bergström and Puri provide explicit constants of proportionality in the $O(\cdot)$ terms, the present development is in closer touch with Hájek (1968), provides some alternative arguments of proof, and provides explicit application to relax the conditions of a theorem of Jurečková and Puri (1975) giving the above rate for the case of location-shift alternatives.

1. Introduction and main results. Hájek (1968) established the asymptotic normality of simple linear rank statistics under broad conditions on the regression constants, the distribution functions of individual observations and the scores-generating function. Corresponding to his theorems for the case of smooth and bounded scores, the rate of convergence $O(N^{-\frac{1}{2}+\delta})$, $N \to \infty$, for any $\delta > 0$ is obtained in the present paper (Theorems 2 and 3). Recently, Bergström and Puri (1977) have already established a version of part of Theorem 2, exhibiting explicit constants in the $O(\cdot)$ term. Previously, Jurečková and Puri (1975) have shown the above rate for the cases (a) iden-

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tical distributions, and (b) location-shift alternatives. In case (a), their score-smoothness condition is slightly milder than that of the present results. However, for case (b) their stringent conditions on the shift parameters and on the score-smoothness are considerably reduced here (Corollary).

Our method of proof consists in approximating the simple linear rank statistic by a sum of independent random variables and establishing, for arbitrary ν, a suitable bound on the ν-th moment of the error of approximation (Theorem 1).

Let $X_{N1}, \ldots, X_{NN}$ be independent random variables with ranks $R_{N1}, \ldots, R_{NN}$. The simple linear rank statistic to be considered is

$$S_N = \sum_{i=1}^{N} c_{Ni} a_N(R_{Ni}), \tag{1.1}$$

where $c_{N1}, \ldots, c_{NN}$ are arbitrary "regression constants" and $a_N(1), \ldots, a_N(N)$ are "scores". Throughout, the following condition will be assumed.

CONDITION A. (i) The scores are generated by a function $\phi(t)$, $0 < t < 1$, in either of the following ways:

$$a_N(i) = \phi\left(\frac{i}{N+1}\right), \ 1 \le i \le N, \tag{1.2}$$

$$a_N(i) = E\phi(U_N^{(i)}), \ 1 \le i \le N, \tag{1.3}$$

where $U_N^{(i)}$ denotes the i-th order statistic in a sample of size N from the uniform distribution on (0, 1).

(ii) ϕ has a bounded second derivative.

(iii) The regression constants satisfy

$$\sum_{i=1}^{N} c_{Ni} = 0, \ \sum_{i=1}^{N} c_{Ni}^2 = 1, \tag{1.4}$$

$$\max_{1 \le i \le N} c_{Ni}^2 = O(N^{-1} \log N), \ N \to \infty. \ \square \tag{1.5}$$

Note that (1.4) may be assumed without loss of generality.

The X_{Ni}'s are assumed to have continuous distribution functions F_{Ni}, $1 \le i \le N$. Put $H_N(x) = N^{-1} \sum_{i=1}^N F_{Ni}(x)$. The derivatives of ϕ will be denoted by ϕ', ϕ'', etc. Also, put $\mu_\phi = \int_0^1 \phi(t)dt$ and $\sigma_\phi{}^2 = \int_0^1 [\phi(t) - \mu_\phi]^2 dt$. Finally, denote by Φ the standard normal cdf. Hereafter the suffix N will be omitted from X_{Ni}, R_{Ni}, c_{Ni}, S_N, F_{Ni}, H_N and other notation.

The statistic S will be approximated by the same sum of independent random variables introduced by Hájek (1968), namely

$$T = \sum_{i=1}^N \ell_i(X_i), \tag{1.6}$$

where

$$\ell_i(x) = N^{-1} \sum_{j=1}^N (c_j - c_i) \int [u(y-x) - F_i(y)] \phi'(H(y)) dF_j(y), \tag{1.7}$$

with

$$u(x) = 1,\ x \ge 0;\ u(x) = 0,\ x < 0. \tag{1.8}$$

THEOREM 1. *Assume Condition A. Then, for every integer* r, *there exists a constant* $M = M(\phi, r)$ *such that*

$$E(S - ES - T)^{2r} \le MN^{-r},\ \textit{all}\ N. \tag{1.9}$$

The case r = 1 was proved by Hájek (1968). The extension to higher order is needed for the present purposes.

THEOREM 2. *Assume Condition A.* (i) *If*

$$\text{Var } S > B > 0,\ N \to \infty, \tag{1.10}$$

then for every $\delta > 0$,

$$\sup_x |P\{S - ES < x(\text{Var } S)^{\frac{1}{2}}\} - \Phi(x)| = O(N^{-\frac{1}{2}+\delta}),\ N \to \infty. \tag{1.11}$$

(ii) *The assertion remains true with* Var S *replaced by* Var T.

(iii) *Both assertions remain true with* ES *replaced by*

$$\mu = \sum_{i=1}^{N} c_i \int \phi(H(x))dF_i(x). \tag{1.12}$$

Compare Theorem 2.1 of Hájek (1968) and Theorem 1.2 of Bergström and Puri (1977).

THEOREM 3. *Assume Condition A and that*

$$\sup_{i,j,x} |F_i(x) - F_j(x)| = O(N^{-\frac{1}{2}} \log N), \ N \to \infty. \tag{1.13}$$

Then for every $\delta > 0$

$$\sup_x |P\{S - ES < x\sigma_\phi\} - \Phi(x)| = O(N^{-\frac{1}{2}+\delta}), \ N \to \infty. \tag{1.14}$$

The assertion remains true with σ_ϕ^2 *replaced by either* Var S *or* Var T, *and/or* ES *replaced by* μ.

Compare Theorem 2.2 of Hájek (1968). As a corollary of Theorem 3, the case of local location-shift alternatives will be treated. The following condition will be assumed.

CONDITION B. (i) The cdf's F_i are generated by a cdf F as follows: $F_i(x) = F(x - \Delta d_i)$, $1 \le i \le N$, with $\Delta \ne 0$.

(ii) F has a density f with bounded derivative f'.

(iii) The shift coefficients satisfy

$$\sum_{i=1}^{N} d_i = 0, \ \sum_{i=1}^{N} d_i^2 = 1, \tag{1.15}$$

$$\max_{1 \le i \le N} d_i^2 = O(N^{-1} \log N), \ N \to \infty. \ \square \tag{1.16}$$

Note that (1.15) may be assumed without loss of generality.

COROLLARY. *Assume Conditions A and B and that*

(1.17) $$\sum_{i=1}^{N} c_i^{\,2} d_i^{\,2} = O(N^{-1} \log N),\ N \to \infty.$$

Then for every $\delta > 0$

(1.18) $$\sup_x |P\{S - \tilde{\mu} < x\sigma_\phi\} - \Phi(x)| = O(N^{-\frac{1}{2}+\delta}),\ N \to \infty,$$

where

(1.19) $$\tilde{\mu} = \Delta(\sum_{i=1}^{N} c_i d_i) \int \phi'(F(x)) f^2(x) dx.$$

(The corresponding result of Jurečková and Puri (1975) requires ϕ to have four bounded derivatives and requires further conditions on the c_i's and d_i's. On the other hand, their result for the case of all F_i's identical requires only a single bounded derivative for ϕ.)

2. The proofs. The main development will be carried out for the case of scores given by (1.2). In Lemma 7 it will be shown that the case of scores given by (1.3) may be reduced to this case.

Assuming ϕ'' bounded, put

(2.1) $$K_1 = \sup_{0<t<1} |\phi'(t)|,\ K_2 = \sup_{0<t<1} |\phi''(t)|.$$

By Taylor expansion the statistic S may be written as

$$S = U + V + W,$$

where, with $\rho_i = R_i/(N+1)$, $1 \le i \le N$,

(2.2) $$U = \sum_{i=1}^{N} c_i \phi(E(\rho_i | X_i)),$$

(2.3) $$V = \sum_{i=1}^{N} c_i \phi'(E(\rho_i | X_i))[\rho_i - E(\rho_i | X_i)]$$

and

(2.4) $$W = \sum_{i=1}^{N} c_i\, K_2\, \xi_i [\rho_i - E(\rho_i | X_i)]^2,$$

the random variables ξ_i satisfying $|\xi_i| \le 1$, $1 \le i \le N$. It will first be shown that W may be neglected. To see this, note that

$$R_i = \textstyle\sum_{j=1}^{N} u(X_i - X_j),\ 1 \le i \le N, \tag{2.5}$$

where $u(\cdot)$ is given by (1.8). Thus

$$E(\rho_i|X_i) = [\textstyle\sum_{\substack{j=1\\ j\ne i}}^{N} F_j(X_i) + 1]/(N+1) \tag{2.6}$$

and

$$\rho_i - E(\rho_i|X_i) = \frac{1}{N+1}\textstyle\sum_{\substack{j=1\\ j\ne i}}^{N} [u(X_i - X_j) - F_j(X_i)]. \tag{2.7}$$

Observe that, given X_i, the summands in (2.7) are conditionally independent random variables centered at means. Hence the following classical result, due to Marcinkiewicz and Zygmund (1937), is applicable.

LEMMA 1. *Let* $Y_1, Y_2, \ldots$ *be independent random variables with mean* 0. *Let* ν *be an integer. Then*

$$E|\sum_{i=1}^{N} Y_i|^{\nu} \le A_{\nu}\, n^{\frac{1}{2}\nu - 1} \sum_{i=1}^{n} E|Y_i|^{\nu}, \tag{2.8}$$

where A_{ν} *is a universal constant depending only on* ν.

LEMMA 2. *Assume* (1.4). *For each positive integer* r,

$$E\, W^{2r} \le K_2^{\,2r} A_{4r} N^{-r},\ all\ N. \tag{2.9}$$

PROOF. Write W in the form $W = K_2 \sum_{i=1}^{N} c_i W_i$. By the Cauchy-Schwarz inequality and (1.4),

$$W^{2r} \le K_2^{\,2r} (\textstyle\sum_{i=1}^{N} c_i^{\,2})^r\, (\sum_{i=1}^{N} W_i^{\,2})^r = K_2^{\,2r}\, (\sum_{i=1}^{N} W_i^{\,2})^r. \tag{2.10}$$

Minkowski's inequality then yields

(2.11) $$E\,W^{2r} \le K_2^{\,2r}\,[\textstyle\sum_{i=1}^{N} (E\,W_i^{\,2r})^{1/r}]^r.$$

By Lemma 1,

(2.12) $$E\{[\rho_i - E(\rho_i|X_i)]^{4r}|X_i\} \le (N+1)^{-4r}\,A_{4r}\,(N-1)^{2r-1}N$$

so that

(2.13) $$E\,W_i^{\,2r} \le E[\rho_i - E(\rho_i|X_i)]^{4r} \le A_{4r}\,N^{-2r}.$$

Thus (2.9) follows. □

Thus S may be replaced by $Z = U + V$, in the sense that $E(S-Z)^{2r} = O(N^{-r})$, $N \to \infty$, each r. It will next be shown that, in turn, Z may be replaced in the same sense by a sum of independent random variables, namely by its *projection*

(2.14) $$\hat{Z} = \textstyle\sum_{i=1}^{N} E(Z|X_i) - (N-1)\,E(Z).$$

Clearly, $\hat{Z} = \hat{U} + \hat{V}$ and $\hat{U} = U$. Thus $Z - \hat{Z} = V - \hat{V}$.

LEMMA 3. *The projection of* V *is*

(2.15) $$\hat{V} = \frac{1}{N+1} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\ne i}}^{N} c_j \ell_{ji}(X_i),$$

where

(2.16) $$\ell_{ji}(x) = \int [u(y-x) - F_i(y)]\phi'(E(\rho_j|X_j=y))dF_j(y).$$

PROOF. Put

$$Y_{ij} = \phi'(E(\rho_i|X_i))[u(X_i-X_j) - F_j(X_i)].$$

For $j \ne i$, $j \ne k$, we have

$$\begin{aligned} E(Y_{ij}|X_k) &= E\{E[Y_{ij}|X_i, X_k]|X_k\} \\ &= E\{\phi'(E(\rho_i|X_i))E[u(X_i-X_j)-F_j(X_i)|X_i, X_k]|X_k\} \qquad (2.17)\\ &= E\{\phi'(E(\rho_i|X_i)) \cdot 0\} = 0. \end{aligned}$$

For $j \neq i$,

(2.18) $$E(Y_{ij}|X_j) = \ell_{ij}(X_j),$$

where $\ell_{mn}(x)$ is defined by (2.16). Also, by (2.17),

(2.19) $$EY_{ij} = 0, \text{ if } i \neq j.$$

Therefore, the projection of Y_{ij}, for $i \neq j$, is $\hat{Y}_{ij} = \ell_{ij}(X_j)$. Since, by (2.3) and (2.7),

(2.20) $$V = \frac{1}{N+1} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} c_j Y_{ji},$$

the projection $\hat{V}$ is given by (2.15). ∎

LEMMA 4. *Assume* (1.4). *For each positive integer* r, *there exists a constant* B_r *such that*

(2.21) $$E(V-\hat{V})^{2r} \leq K_1^{2r} B_r N^{-r}, \textit{ all } N.$$

PROOF. By (2.15) and (2.20),

(2.22) $$E(V-\hat{V})^{2r} = (N+1)^{-2r} \sum_{i_1=1}^{N} \cdots \sum_{i_{2r}=1}^{N} c_{i_1} \cdots c_{i_{2r}} \sum_{\substack{j_1=1 \\ j_1\neq i_1}}^{N} \cdots \sum_{\substack{j_{2r}=1 \\ j_{2r}\neq i_{2r}}}^{N} \delta_{i_1, j_1, \ldots, i_{2r}, j_{2r}},$$

where

(2.23) $$\delta_{i_1, j_1, \ldots, i_{2r}} = E \prod_{k=1}^{2r} [Y_{i_k j_k} - \ell_{i_k j_k}(X_{j_k})].$$

Consider a typical term of the form (2.23). If the i_k index in the k-th factor occurs only in that factor, then the entire product of factors has expectation 0, for

$$E[Y_{i_k j_k} - \ell_{i_k j_k}(X_{j_k}) | X_m, m \in \{i_1, \dots, i_{k-1}, i_{k+1}, \dots,$$

$$i_{2r}, j_1, \dots, j_{2r}\}] = 0.$$

By a similar argument, the same conclusion holds if the j_k index in the k-th factor occurs only in that factor. Thus the expectation in (2.23) is possibly nonzero only if each factor has both indices repeated in other factors. Among such cases, consider now only those terms corresponding to a given pattern of the possible identities $i_a = i_b$, $i_a = j_b$, $j_a = j_b$ for $1 \le a \le 2r$, $1 \le b \le 2r$. For example, for $r = 3$, one such specific pattern is: $i_2 = i_1$, $i_3 \ne i_1$, $i_4 = i_1$, $i_5 = i_3$, $i_6 \ne i_1$, $i_6 \ne i_3$, $j_2 = j_1$, $j_3 = j_1$, $j_4 \ne j_1$, $j_5 = j_4$, $j_6 = j_4$, $j_1 = i_3$, $j_4 \ne i_1$. In general, there are at most 2^{6r} such patterns. For such a pattern, let q denote the number of distinct values among $i_1, \dots, i_{2r}$ and p the number of distinct values among $j_1, \dots, j_{2r}$. Let p_1 denote the number of distinct values among $j_1, \dots, j_{2r}$ *not* appearing among $i_1, \dots, i_{2r}$ and put $p_2 = p - p_1$. Within the given constraints, and after selection of $i_1, \dots, i_{2r}$, the number of choices for $j_1, \dots, j_{2r}$ clearly is of order

$$O(N^{p_1}). \tag{2.24}$$

Now clearly $2p_1 \le 2r - p_2$, i.e.,

$$p_1 \le r - \tfrac{1}{2}p_2. \tag{2.25}$$

Now let q_1 denote the number of $i_1, \dots, i_{2r}$ used only *once* among $i_1, \dots, i_{2r}$. Then obviously

$$q_1 \le p_2. \tag{2.26}$$

By (2.24), (2.25) and (2.26), it is seen that the contribution to (2.22) from summation over $j_1, \ldots, j_{2r}$ is of order at most

$$O(N^{r-\frac{1}{2}q_1}),$$

since the quantity in (2.23) is of magnitude $\le K_1^{2r}$. It follows that

$$E(V-\hat{V})^{2r} \le (N+1)^{-2r} K_2^{2r} [O(N^{r-\frac{1}{2}q_1})] \sum_{\ell_1=1}^{N} \cdots \sum_{\ell_q=1}^{N} |c_{\ell_1}^{a_1} \cdots c_{\ell_q}^{a_q}|, \tag{2.27}$$

where $a_1, \ldots, a_q$ are integers satisfying $a_i \ge 1$, $a_1 + \ldots + a_q = 2r$, and exactly q_1 of the a_i's are equal to 1. Now, for $a \ge 2$,

$$\textstyle\sum_{i=1}^{N} |c_i|^a \le (\sum_{i=1}^{N} c_i^2)^{\frac{1}{2}a} = 1, \tag{2.28}$$

by (1.4). Further,

$$\textstyle\sum_{i=1}^{N} |c_i| \le N(\frac{1}{N} \sum_{i=1}^{N} c_i^2)^{\frac{1}{2}} = N^{\frac{1}{2}}. \tag{2.29}$$

Thus

$$\sum_{\ell_1=1}^{N} \cdots \sum_{\ell_q=1}^{N} |c_{\ell_1}^{a_1} \cdots c_{\ell_q}^{a_q}| \le N^{\frac{1}{2}q_1}. \tag{2.30}$$

Combining (2.27) and (2.30), we obtain (2.21). □

Next it is shown that Z may be replaced by $\tilde{Z} = \tilde{U} + \tilde{V}$, where

$$\tilde{U} = \textstyle\sum_{i=1}^{N} c_i \, \phi(H(X_i)) \tag{2.31}$$

and

$$\tilde{V} = \frac{1}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\ne i}}^{N} c_j \, \tilde{\ell}_{ji}(X_i), \tag{2.32}$$

with

$$\tilde{\ell}_{ji}(x) = \int [u(y-x) - F_i(y)] \phi'(H(y)) dF_j(y). \tag{2.33}$$

LEMMA 5. *Assume* (1.4). *Then*

$$|\hat{Z} - \tilde{Z}| \le (K_2 + 3K_1)N^{-\frac{1}{2}}. \tag{2.34}$$

PROOF. By (2.6),

$$E(\rho_i|X_i) = H(X_i) + \frac{1-F_i(X_i)-H(X_i)}{N+1}. \tag{2.35}$$

Hence, by the Mean Value Theorem,

$$|\phi(E(\rho_i|X_i)) - \phi(H(X_i))| \le K_1N^{-1}. \tag{2.36}$$

Therefore, by (2.29),

$$|U - \tilde{U}| \le K_1N^{-\frac{1}{2}}. \tag{2.37}$$

Now

$$\hat{V} - \tilde{V} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} c_j[\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)] - \frac{1}{N}\hat{V} - \frac{1}{N} \sum_{i=1}^{N} c_i\ell_{ii}(X_i).$$

But

$$|\hat{V}| \le K_1 \textstyle\sum_{i=1}^{N} |c_i| \le K_1N^{\frac{1}{2}},$$

i.e.,

$$\frac{1}{N} |\hat{V}| \le K_1N^{-\frac{1}{2}}. \tag{2.38}$$

Similarly,

$$|\frac{1}{N} \textstyle\sum_{i=1}^{N} c_i\ell_{ii}(X_i)| \le K_1N^{-\frac{1}{2}}. \tag{2.39}$$

Finally,

$$|\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)| \le K_2N^{-1}$$

so that

$$(2.40) \qquad \left|\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{N}c_j[\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)]\right| \le K_2N^{-\frac{1}{2}}.$$

Thus (2.34) follows. □

Now we connect with the random variable T of Theorem 1.

LEMMA 6. *Let* T *be defined by* (1.6) *and* μ *by* (1.12). *Then*

$$(2.41) \qquad \tilde{Z} - \mu = T$$

and there exists a constant $K_4 = K_4(\phi)$ *such that*

$$(2.42) \qquad |ES - \mu| \le K_4N^{-\frac{1}{2}}.$$

PROOF. (2.42) is shown by Hájek (1968), p. 340. To obtain (2.41), check that

$$(2.43) \qquad \begin{aligned}\tilde{Z} - \mu - T = \sum_{i=1}^{N} c_i\{&\phi(H(X_i)) - E\phi(H(X_i)) \\ &+ \int[u(x-X_i) - F_i(x)]\phi'(H(x))dH(x)\}.\end{aligned}$$

Now, by integration by parts, for any distribution function G we have

$$\int\phi'(H(x))G(x)dH(x) = -\int\phi(H(x))dG(x) + \text{constant},$$

where the constant may depend on ϕ and $H(\cdot)$ but not on $G(\cdot)$. Thus the sum in (2.43) reduces to 0. □

Up to this point, only the scores given by (1.2) have been considered. The next result provides the basis for interchanging with the scores given by (1.3).

LEMMA 7. *Denote* $\sum_{i=1}^{N} c_ia_N(R_i)$ *by* S *in the case corresponding to* (1.2) *and by* S' *in the case corresponding to* (1.3). *Assume* (1.4). *Then there exists* $K_5 = K_5(\phi)$ *such that*

$$|S - ES - (S' - ES')| \le K_5 N^{-\frac{1}{2}}. \tag{2.43}$$

PROOF. It is easily found (see Hájek (1968), p. 341) that

$$\left|\phi\left(\frac{i}{N+1}\right) - E\phi(U_N^{(i)})\right| \le K_0 N^{-1}, \tag{2.44}$$

where K_0 does not depend on i or N. Thus, by (2.29),

$$|S - S'| \le K_0 N^{-\frac{1}{2}} \tag{2.45}$$

and hence also

$$|ES - ES'| \le K_0 N^{-\frac{1}{2}}. \tag{2.46}$$

Thus (2.43) follows with $K_5 = 2K_0$. □

PROOF OF THEOREM 1. Consider first the case (1.2). By Minkowski's inequality,

$$\begin{aligned}[E(S - ES - T)^{2r}]^{1/2r} &\le [E(S - Z)^{2r}]^{1/2r} + [E(Z - \hat{Z})^{2r}]^{1/2r} \\ &\quad + [E(\hat{Z} - \tilde{Z})^{2r}]^{1/2r} + [E(\tilde{Z} - \mu - T)^{2r}]^{1/2r} \\ &\quad + |ES - \mu|.\end{aligned} \tag{2.47}$$

By Lemmas 2, 4, 5 and 6, each term on the right-hand side of (2.47) may be bounded by $KN^{-\frac{1}{2}}$ for a constant $K = K(\phi, r)$ depending only on ϕ and r. Thus (1.9) follows. In the case of scores given by (1.3), we combine Lemma 7 with the preceding argument. □

PROOF OF THEOREM 2. First assertion (i) will be proved. Put

$\alpha_N = \sup_x|P\{S - ES < x(\mathrm{Var}\ S)^{\frac{1}{2}}\} - \Phi(x)|$,

$\beta_N = \sup_x|P\{T < x(\mathrm{Var}\ S)^{\frac{1}{2}}\} - \Phi(x)|$,

and

$$\gamma_N = \sup_x |P\{T < x(\mathrm{Var}\ T)^{\frac{1}{2}}\} - \Phi(x)|.$$

By a standard device, if

(2.48) $$\beta_N = O(a_N),\ N \to \infty,$$

for a sequence of constants $\{a_N\}$, then

(2.49) $$\alpha_N = O(a_N) + P\{|S - ES - T|/(\mathrm{Var}\ S)^{\frac{1}{2}} > a_N\},\ N \to \infty.$$

We shall obtain a condition of form (2.48) by first considering γ_N. By the classical Berry-Esséen theorem, as stated in Loève (1963), p. 288,

(2.50) $$\gamma_N \le C(\mathrm{Var}\ T)^{-3/2} \sum_{i=1}^N E|\ell_i(X_i)|^3,$$

where C is a universal constant. Clearly,

$$|\ell_i(X_i)| \le K_1 N^{-1} \sum_{j=1}^N |c_j - c_i|.$$

Now

(2.51) $$\left(\sum_{j=1}^N |c_j - c_i|\right)^2 \le N\left[\sum_{j=1}^N (c_j - c_i)^2\right] = N[1 + Nc_i^{\,2}].$$

By the elementary inequality (Loève (1963), p. 155)

(2.52) $$|x+y|^m \le \theta_m |x|^m + \theta_m |y|^m,$$

where $m > 0$ and $\theta_m = 1$ or 2^{m-1} according as $m \le 1$ or $m \ge 1$, we thus have

$$\left(\sum_{j=1}^N |c_j - c_i|\right)^3 \le N^{3/2}\, 2^{\frac{1}{2}}\, (1 + N^{3/2}|c_i|^3)$$

and hence

(2.53) $$\sum_{i=1}^N E|\ell_i(X_i)|^3 \le 2^{\frac{1}{2}} K_1^{\,3} \left[N^{-\frac{1}{2}} + \sum_{i=1}^N |c_i|^3\right].$$

Now, by a double application of (2.52),

$$|\text{Var } S - \text{Var } T| = |E(S - ES - T)(S - ES + T)|$$

$$(2.54) \qquad \le [E(S - ES - T)^2]^{\frac{1}{2}} [2 \text{ Var } S + 2 \text{ Var } T]^{\frac{1}{2}}$$

$$\le [E(S - ES - T)^2]^{\frac{1}{2}} \sqrt{2} [(\text{Var } S)^{\frac{1}{2}} + (\text{Var } T)^{\frac{1}{2}}].$$

Writing

$$|(\text{Var } S)^{\frac{1}{2}} - (\text{Var } T)^{\frac{1}{2}}| = \frac{|\text{Var } S - \text{Var } T|}{(\text{Var } S)^{\frac{1}{2}} + (\text{Var } T)^{\frac{1}{2}}}$$

and applying Theorem 1 in conjunction with (2.54), we have

$$(2.55) \qquad |(\text{Var } S)^{\frac{1}{2}} - (\text{Var } T)^{\frac{1}{2}}| \le M_0 N^{-\frac{1}{2}},$$

where the constant M_0 depends only on ϕ. It follows that if Var S is bounded away from 0, as per assumption (1.10), then the same holds for Var T, and conversely. Consequently, by (1.10), (2.50), (2.53) and (2.55), we have

$$\gamma_N' = O(N^{-\frac{1}{2}}) + O(\textstyle\sum_{i=1}^{N} |c_i|^3), \; N \to \infty.$$

Therefore, by (1.4) and (1.5),

$$(2.56) \qquad \gamma_N = O(N^{-\frac{1}{2}} \log N), \; N \to \infty.$$

Now it is easily seen that

$$(2.57) \qquad \beta_N \le \gamma_N + O\left(\left|\frac{(\text{Var } S)^{\frac{1}{2}}}{(\text{Var } T)^{\frac{1}{2}}} - 1\right|\right).$$

By (1.10) and (2.55), the right-most term in (2.57) is $O(N^{-\frac{1}{2}})$. Hence

$$(2.58) \qquad \beta_N = O(N^{-\frac{1}{2}} \log N).$$

Therefore, for any sequence of constants a_N satisfying $N^{-\frac{1}{2}} \log N = O(a_N)$, we have (2.48) and thus (2.49). A further application of Theorem 1, with Markov's inequality, yields for arbitrary r

$$(2.59) \qquad P\{|S - ES - T|/(\text{Var } S)^{\frac{1}{2}} > a_N\} \leq a_N^{-2r} (\text{Var } S)^{-r} MN^{-r}.$$

Hence (2.49) becomes

$$\alpha_N = O(a_N) + O(a_N^{-2r} N^{-\frac{1}{2}}).$$

Choosing $a_N = O(N^{-r/(2r+1)})$, we obtain

$$(2.60) \qquad \alpha_N = O(N^{-r/(2r+1)}), \; N \to \infty.$$

Since (2.60) holds for arbitrarily large r, the first assertion of Theorem 2 is established.

Assertions (ii) and (iii) are obtained easily from the foregoing arguments. □

PROOF OF THEOREM 3. It is shown by Hájek (1968), p. 342, that

$$(2.61) \qquad |(\text{Var } T)^{\frac{1}{2}} - \sigma_\phi| \leq 2^{\frac{1}{2}}(K_1 + K_2) \sup_{i,j,x} |F_i(x) - F_j(x)|.$$

The proof is now straightforward using the arguments of the preceding proof. □

PROOF OF THE COROLLARY. By Taylor expansion,

$$(2.62) \qquad |F_i(x) - F(x) -(-\Delta d_i f(x))| \leq A\, \Delta^2 d_i^{\,2},$$

where A is a constant depending only on F. Hence, by (1.15) and (1.16),

$$(2.63) \qquad \sup_{i,j,x} |F_i(x) - F_j(x)| = O(\max_i |d_i|) = O(N^{-\frac{1}{2}} \log N),$$

so that the hypothesis of Theorem 3 is satisfied. It remains to show that ES

may be replaced by the more convenient parameter $\tilde{\mu}$. A further application of (2.62), with (1.15), yields

$$|H(x) - F(x)| \le A\ \Delta^2 N^{-1}$$

so that

$$|\phi(H(x)) - \phi(F(x))| \le K_1 A\ \Delta^2 N^{-1}.$$

Hence, by (2.29),

$$|\mu - \textstyle\sum_{i=1}^N c_i \int \phi(F(x))dF_i(x)| \le K_1 A\ \Delta^2 N^{-\frac{1}{2}}. \tag{2.64}$$

By integration by parts, along with (1.4) and (2.62),

$$\begin{aligned}\textstyle\sum_{i=1}^N c_i \int \phi(F(x))dF_i(x) &= -\textstyle\sum_{i=1}^N c_i \int F_i(x)\phi'(F(x))dF(x) \\ &= -\textstyle\sum_{i=1}^N c_i \int (-\Delta d_i) f(x)\phi'(F(x))dF(x) + \eta\ A\ \Delta^2 \textstyle\sum_{i=1}^N c_i {d_i}^2, \qquad (2.65) \\ &= \tilde{\mu} + \eta\ A\ \Delta^2\ \textstyle\sum_{i=1}^N c_i {d_i}^2,\end{aligned}$$

where $|\eta| \le 1$. Now, by (1.15),

$$\textstyle\sum_i |c_i| {d_i}^2 \le (\sum_i {c_i}^2 {d_i}^2)^{\frac{1}{2}}. \tag{2.66}$$

By (2.64), (2.65), (2.66) and (1.17),

$$|\mu - \tilde{\mu}| = O(N^{-\frac{1}{2}} \log N),\ N \to \infty.$$

Thus μ may be replaced by $\tilde{\mu}$ in Theorem 3. □

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to the power $-\frac{1}{2}$+delta

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The rate of convergence $O(N^{-\frac{1}{2}+\delta})$ for any $\delta > 0$ is established for two theorems of Hájek (1968) on asymptotic normality of simple linear rank statistics. These pertain to smooth and bounded scores, arbitrary regression constants, and broad conditions on the distributions of individual observations. The results parallel those of Bergström and Puri (1977), which appeared in print just as this paper was completed. Whereas Bergström and Puri provide explicit constants of proportionality in the $O(\cdot)$ terms, the present development is in closer touch with Hájek (1968), provides some alternative arguments of proof, and provides explicit application to relax the conditions of a theorem of Jurečková and Puri (1975) giving the above rate for the case of location-shift alternatives.